PROPAGATION OF A SIGNAL IN A LIQUID WITH A CONTINUOUS DISTRIBUTION OF BUBBLE SIZES

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The theory of wave propagation in a liquid with gas bubbles based on the equations of [1-3] has been worked out mainly for the case of monodispersed bubbles (bubbles of only one size exist at each point in space). Reviews of the results can be found in [4-6]. Fewer papers have been devoted to liquids with a distribution of bubble sizes. As a rule, these papers assume a discrete size distribution function for the bubbles (mixture of bubbles with a finite number of fractions) [7-13]. The case of a continuous size distribution function for the bubbles was considered qualitatively in [14-18]. An asymptotic solution of the Cauchy problem for the linearized equations of motion of a liquid with a continuous spectrum of bubbles was constructed in [19]. In particular, it was shown that a monochromatic wave with frequency equal to one of the natural frequencies of the bubbles cannot propagate in the liquid. The wave splits up into a sum of two monochromatic waves (low and high-frequency components) or is transformed into a high-frequency wave, depending on whether the wavelength of the incident wave is sufficiently large or small. The present paper is a continuation of [19]. For completeness we rederive the integrodifferential equation given in [19]. This equation is then used to study the time evolution of a signal propagating in a medium which is initially at rest.

1. Equations of Motion. The nonlinear equations of motion of a liquid with bubbles have the form [1-3, 14, 18]

$$\begin{gathered}\frac{d\rho}{dt}+\rho\operatorname{div}\mathbf{u}=0,\quad \frac{d\mathbf{u}}{dt}+\frac{\nabla p}{\rho}=0,\quad \frac{d}{dt}=\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla,\\ R\frac{d^2R}{dt^2}+\frac{3}{2}\left(\frac{dR}{dt}\right)^2=\frac{1}{\rho_{l0}}\left(p_0\left(\frac{\xi}{R}\right)^{3\gamma}-p\right),\\ dN/dt+N\operatorname{div}\mathbf{u}=0,\quad \rho=\rho_l(1-\alpha_2),\quad p=p_0+c_{l0}^2(\rho_l-\rho_{l0}),\\ \alpha_2=\frac{4}{3}\pi\int_0^\infty N(\xi,t,\mathbf{x})R^3(\xi,t,\mathbf{x})\,d\xi.\end{gathered}\tag{1.1}$$

Here t is the time; $\mathbf{x}$ is the spatial coordinate; $\rho(t,\mathbf{x})$ is the density of the mixture $\rho_l(t,\mathbf{x})$ is the density of the liquid; $\mathbf{u}(t,\mathbf{x})$ is the velocity of the mixture; ξ is the radius of a bubble of "fraction ξ" in equilibrium (ξ lies within $[\xi_1, \xi_2]$, $0 < \xi_1 < \xi_2 < \infty$ and plays the role of a Lagrangian variable determining the corresponding fraction of bubbles); $R(\xi, t, \mathbf{x})$ is the bubble radius; $p(t,\mathbf{x})$ is the pressure of the mixture, which is assumed to be equal to the pressure of the liquid; c_{l0} is the speed of sound in the liquid; $N(\xi,t,\mathbf{x})$ is the size distribution function of the bubbles, defined such that $dn=N(\xi,t,\mathbf{x})d\xi$ is the number of bubbles of fraction ξ per unit volume of the mixture; $\alpha_2(t,\mathbf{x})$ is the volume concentration of bubbles. A zero subscript corresponds to the state of rest.

The system of equations (1.1) contains an infinite number of variables, characterized by the index ξ. Some of these variables depend only on t, $\mathbf{x}$, while others depend on t, $\mathbf{x}$, ξ. If $N=N_*(t,\mathbf{x})\delta(\xi-\xi_*)$, where δ is the delta function and ξ_* is a constant bubble radius, then (1.1) reduces to the equation of motion of a monodispersed mixture.

The equation of continuity and the conservation of number density of the bubbles give the relation

$$\frac{d}{dt}\left(\frac{N}{\rho}\right)=0.\tag{1.2}$$

Assuming that at t = 0 the distribution of bubbles is independent of $\mathbf{x}$, we obtain from (1.2)

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$$N/\rho = N_0(\xi)/\rho_0, \quad \rho_0 = \rho_{l0}\left(1 - \frac{4}{3}\pi\int_0^\infty N_0(\xi)\xi^3 d\xi\right), \tag{1.3}$$

where $N_0(\xi)$ is the distribution of bubbles in equilibrium. It is assumed that $N_0(\xi)$ is finite on $[0, \infty)$ and its argument is inside $I = [\xi_1, \xi_2]$. Solving (1.3) for $N(\xi, t, \mathbf{x})$ in terms of $N_0(\xi)$, the density of the mixture is given by

$$\rho = \rho_l\left(1 - \frac{\rho_l}{\rho_0}\frac{4}{3}\pi\int_0^\infty N_0(\xi) R^3(\xi, t, \mathbf{x})\, d\xi\right) + O(\alpha_2^2). \tag{1.4}$$

Let ε_1 be a small parameter. Linearizing (1.1) with the use of (1.4) and the following form of the solution

$$p = p_0 + \varepsilon_1 p', \quad \mathbf{u} = \varepsilon_1 \mathbf{u}', \quad R = \xi + \varepsilon_1 R', \quad \rho_l = \rho_{l0} + \varepsilon_1 p'/c_{l0}^2.$$

after discarding terms of order ε_1^2 and α_2^2 and eliminating the velocity $\mathbf{u}'$, we obtain

$$\begin{gathered} c_f^{-2} p'_{tt} - \Delta p' = 4\pi\rho_{l0}\int_0^\infty N_0(\xi)\xi^2 R'_{tt} d\xi, \\ \xi R'_{tt} + \omega^2(\xi)\xi R' = -p'/\rho_{l0}, \quad \omega^2(\xi) = \varkappa^2/\xi^2, \quad \varkappa^2 = 3\gamma p_0/\rho_{l0}. \end{gathered} \tag{1.5}$$

Here Δ is the Laplacian in $\mathbf{x}$; $c_f = c_{l0}(1 + \alpha_{20})$ is the "frozen" speed of sound; α_{20} is the volume concentration of bubbles in equilibrium; $\omega(\xi)$ is the natural frequency of vibration of bubbles of type ξ. The system of equations (1.5) is practically the same as in [15], but c_f takes the place of c_{l0}. It is obviously more correct in this approximation to assume that the speed of sound in the pure liquid is equal to the frozen speed of sound. Nevertheless, below we use the traditional notation c_f. Omitting the primes and performing the substitution $A = \xi R$, we finally obtain

$$c_f^{-2} p_{tt} - \Delta p = 4\pi\rho_{l0}\int_0^\infty N_0(\xi)\xi A_{tt} d\xi, \quad A_{tt} + \omega^2(\xi) A = -p/\rho_{l0}. \tag{1.6}$$

Below we consider the one-dimensional case for simplicity. The initial and boundary conditions for (1.6) for the case of a signal propagating in a liquid at rest can be written in the form

$$\begin{gathered} p|_{t=0} = p_t|_{t=0} = 0, \; A|_{t=0} = A_t|_{t=0} = 0, \; x > 0, \\ p|_{x=0} = \mu(t), \; t > 0. \end{gathered} \tag{1.7}$$

2. Reduction to a Single Equation. We transform from ξ to ω:

$$\begin{gathered} c_f^{-2} p_{tt} - p_{xx} = 4\pi\rho_{l0}\varkappa^2\int_0^\infty \frac{\tilde N_0(\omega)}{\omega^3}\tilde A_{tt} d\omega, \\ \tilde A_{tt} + \omega^2\tilde A = -p/\rho_{l0}, \; \varkappa^2 = 3\gamma p_0/\rho_{l0}. \end{gathered} \tag{2.1}$$

Here $\tilde N_0(\omega) = N_0(\varkappa/\omega)$; $\tilde A(\omega, t, x) = A(\varkappa/\omega, t, x)$. The function $\tilde N_0(\omega)$ vanishes outside the interval $J = [\omega_1, \omega_2]$ we have $\omega_1 = \varkappa/\xi_2$, $\omega_2 = \varkappa/\xi_1$. From the second equation of (2.1) it follows that

$$\tilde A = -\frac{1}{\rho_{l0}\omega}\int_0^t p(\tau, x)\sin\omega(t - \tau)\, d\tau. \tag{2.2}$$

Substituting (2.2) into the first equation of (2.1), we finally have

$$c_f^{-2} p_{tt} - p_{xx} + m^2 p = \int_0^t K(t - \tau)\, p(\tau, x)\, d\tau, \tag{2.3}$$

where

$$\begin{gathered} m^2 = \int_0^\infty \frac{N(\omega)}{\omega^3} d\omega, \quad K(t - \tau) = \int_0^\infty \frac{N(\omega)}{\omega^2}\sin\omega(t - \tau)\, d\omega, \\ N(\omega) = 4\pi\varkappa^2\tilde N_0(\omega), \; \varkappa^2 = 3\gamma p_0/\rho_{l0}. \end{gathered} \tag{2.4}$$

Equations (2.3) and (2.4), with the initial conditions (1.7), are the basis for further analysis. The presence of the convolution operator on the right-hand side of (2.3) shows that a liquid with bubbles is a medium with memory. An equation of the type (2.3) was obtained in [20] in the monodispersed case.

3. Propagation of a Signal. We consider the Laplace transform

$$P(s, x) = \int_0^\infty p(t, x) e^{-st} dt, \quad M(s) = \int_0^\infty \mu(t) e^{-st} dt, \quad s = \sigma + i\eta.$$

Then (2.3) with the initial and boundary conditions (1.7) reduces to the following boundary-value problem

$$P_{xx} - L(s) P = 0, \ x > 0, \ P|_{x=0} = M(s), \ P \underset{x\to\infty}{\longrightarrow} 0,$$
$$L(s) = s^2 \left(c_f^{-2} + \int_0^\infty \frac{N(\omega)\, d\omega}{\omega^3(\omega^2 + s^2)} \right). \tag{3.1}$$

Assume that $\mu(t)$ is integrable, it follows that $M(s)$ has no singularities in the right half plane. We assume that $N(\omega)$ obeys the conditions:

A. The function $N(\omega)$ is sufficiently smooth and vanishes identically outside the interval $J = [\omega_1, \omega_2]$; $N(\omega) > 0$ inside J; $N(\omega_i) = 0$, $N'(\omega_i) = 0$, $i = 1, 2$, $0 < \omega_1 < \omega_2 < \infty$.

B. $c_f^{-2} + \lim_{\eta\to\omega_2+0} \int_0^\infty \frac{N(\omega)\, d\omega}{\omega^3(\omega^2 - \eta^2)} < 0$. Since c_f^{-2} is a small quantity, condition B will be satisfied for reasonable distribution functions $N(\omega)$.

It follows from condition A that the function $L(s)$ in (3.1) is analytic in the complex s plane except on the segments $[-i\omega_2, -i\omega_1]$, $[i\omega_1, i\omega_2]$ along the imaginary axis. The quantity $\sqrt{L(s)}$ is understood to be the branch of the square root function which is positive if the radicand is positive. Then the solution of (3.1) has the form

$$P = M(s) \exp(-\sqrt{L(s)}x). \tag{3.2}$$

The function $L(s)$ has three zeros, one of which ($s = 0$) is a double root while the other two simple roots s_k are purely imaginary and complex conjugates. Hence

$$s_1 = i\omega_*, \ s_2 = -i\omega_*, \ c_f^{-2} + \int_0^\infty \frac{N(\omega)\, d\omega}{\omega^3(\omega^2 - \omega_*^2)} = 0.$$

In view of condition B on the function $N(\omega)$ we have $\omega_2 < \omega_* < \infty$. The points $\pm i\omega_*$ are branching points of the function $\sqrt{L(s)}$. We consider the cuts on the complex plane shown in Fig. 1. Using the inversion formula for the Laplace transform, we find from (3.2)

$$p(t, x) = \frac{1}{2\pi i} \lim_{b\to\infty} \int_{\sigma-ib}^{\sigma+ib} M(s) \exp\left(st - \sqrt{L(s)}\, x\right) ds, \ \sigma > 0. \tag{3.3}$$

We consider the ray $x/t = c$ in the (t, x) plane. Our next objective is to obtain the asymptotic form of the solution $p(t, x)$ defined by (3.3) along this ray for large t. We show first that disturbances cannot propagate with a velocity larger than c_f, i.e., in the region $x/t > c_f$ the solution of (2.3) is trivial. Indeed, for large $\mathrm{Re}(s) = \sigma$ and fixed t we have

$$p(t, x) = \frac{1}{2\pi i} \lim_{b\to\infty} \int_{\sigma-ib}^{\sigma+ib} M(s) \exp\left\{\left(1 - \frac{c}{c_f}\right) st + O\left(\frac{1}{s}\right)\right\} ds.$$

We close the segment $[\sigma - ib, \sigma + ib]$ by a circle C_b and denote the enclosed region as D_b and its boundary as ∂D_b (see Fig. 1). Then from Jordan's lemma [21]

$$p(t, x) = \frac{1}{2\pi i} \lim_{b\to\infty} \int_{\partial D_b} M(s) \exp\left\{\left(1 - \frac{c}{c_f}\right) st + O\left(\frac{1}{s}\right)\right\} ds.$$

Because the integrand is analytic, the pressure $p(t, x) \equiv 0$ when $c > c_f$. This result is completely analogous to the result obtained in [22, Chap. 10].

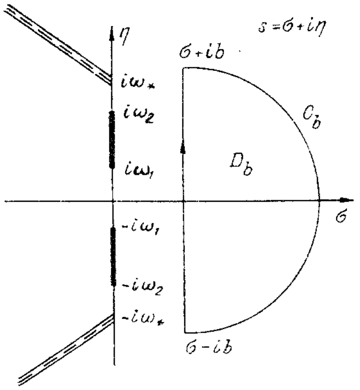

Fig. 1

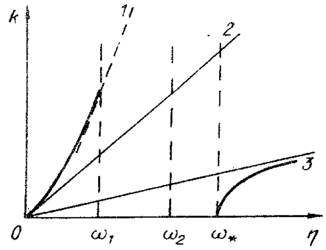

Fig. 2

Suppose now that $c < c_f$. In the inversion formula (3.3) we put $\sigma = 0$. Since the expression for $L(s)$ involves an integral of the Cauchy type, at points belonging to the segments $[-i\omega_2, -i\omega_1]$, $[i\omega_1, i\omega_2]$ it is necessary to calculate the limit of $L(s)$ as s approaches the imaginary axis from the right.† Let $L^-(s)$ be this limiting value. Suppose that $s = \delta + i\Omega$, where $\delta > 0$ and Ω is an arbitrary interior point of the interval $[\omega_1, \omega_2]$ [if Ω is equal to ω_k $(k = 1, 2)$ then $L(s)$ has no singularities when $s \to i\omega_k$, since $N(\omega_k) = 0$]. Then

$$L(s)=s^2\left(c_f^{-2}+\int_{\omega_1}^{\omega_2}\frac{N(\omega)\,d\omega}{\omega^3(\omega^2+s^2)}\right)=s^2\left(c_f^{-2}+\int_{\omega_1}^{\omega_2}\frac{\frac{N(\omega)}{\omega^3}-\frac{N(\Omega)}{\Omega^3}}{\omega^2+s^2}\,d\omega+\frac{N(\Omega)}{\Omega^3 2is}\left\{\int_{\omega_1}^{\omega_2}\frac{d\omega}{\omega+\Omega-i\delta}-\int_{\omega_1}^{\omega_2}\frac{d\omega}{\omega-\Omega+i\delta}\right\}\right).$$

The first two integrals do not have singularities when $\delta \to 0$. The last integral can be evaluated explicitly. Indeed,

$$\int_{\omega_1}^{\omega_2}\frac{d\omega}{\omega-\Omega+i\delta}=\frac{1}{2}\ln\frac{(\omega_2-\Omega)^2+\delta^2}{(\omega_1-\Omega)^2+\delta^2}-i\left\{\operatorname{arctg}\frac{\omega_2-\Omega}{\delta}-\operatorname{arctg}\frac{\omega_1-\Omega}{\delta}\right\}.$$

Then

$$\lim_{\delta\to 0}\int_{\omega_1}^{\omega_2}\frac{d\omega}{\omega-\Omega+i\delta}=\ln\frac{\omega_2-\Omega}{\Omega-\omega_1}-i\pi.$$

Therefore

$$L^-(s)=-\Omega^2\left(c_f^{-2}+\int_{\omega_1}^{\omega_2}\frac{\frac{N(\omega)}{\omega^3}-\frac{N(\Omega)}{\Omega^3}}{\omega^2-\Omega^2}\,d\omega\right)+\frac{N(\Omega)}{2\Omega^2}\left(\ln\frac{\omega_2+\Omega}{\omega_1+\Omega}-\ln\frac{\omega_2-\Omega}{\Omega-\omega_1}+i\pi\right). \tag{3.4}$$

The analogous formula can be obtained when $s \to -i\Omega + 0$, $\Omega \in [\omega_1, \omega_2]$.

If $s \in (-i\omega_*, -i\omega_2) \cup (i\omega_2, i\omega_*)$, then $\sqrt{L(s)} > 0$ and the contribution of these segments to the asymptotic solution is exponentially small. Further, let $s \in (-i\omega_2, -i\omega_1) \cup (i\omega_1, i\omega_2)$. It follows from (3.4) that $\operatorname{Re}(\sqrt{L^-(s)}) > 0$. Hence the contribution of the continuous spectrum is also exponentially small. Hence it is sufficient to consider the following integral for large values of t

$$p(c,t)=\frac{1}{2\pi}\int M(i\eta)\exp(i\Phi(c,\eta)t)\,d\eta,$$
$$\Phi(c,\eta)=\eta-c\sqrt{\eta^2\left(c_f^{-2}+\int_0^\infty\frac{N(\omega)\,d\omega}{\omega^3(\omega^2-\eta^2)}\right)}. \tag{3.5}$$

†In [19] the corresponding limits of the integrals of the Cauchy type were taken incorrectly (Lemma 4.2). This was pointed out to the author by K. V. Lotov. The correct result is obtained in analogy with the present paper and reduces to inserting the factor 1/2 into the formula for the jump of the integrals. This change does not affect the subsequent analysis or conclusions.

In (3.5) the integral is taken along the real axis minus the segments $(-\omega_*, -\omega_1) \cup (\omega_1, \omega_*)$. The integral (3.5) can be evaluated approximately using the method of stationary phase (see [22] for example). The stationary points of the phase of $\Phi(c, \eta)$ are found from the equation

$$\frac{1}{c} = \operatorname{sgn}(\eta) \frac{c_f^{-2} + \int\limits_0^\infty \frac{N(\omega)\, d\omega}{\omega(\omega^2 - \eta^2)^2}}{\sqrt{c_f^{-2} + \int\limits_0^\infty \frac{N(\omega)\, d\omega}{\omega^3(\omega^2 - \eta^2)}}}, \tag{3.6}$$

which is meaningful only when $\eta > 0$. This corresponds to the requirement that the solution should represent a wave traveling to the right. The relation (3.6) has a simple interpretation: if η is a given frequency, then c is the group velocity (a more detailed discussion is given in [19]). The dispersion curve (the dependence of the wave number k on frequency η) constructed in [19] has the form shown in Fig. 2. The curves $k(\eta)$ corresponding to the high and low-frequency branches of the dispersion relation increase monotonically and are convex. In addition

$$\lim_{\eta \to \omega_1 - 0} \frac{dk}{d\eta} = \frac{1}{c_{cr}} < \infty, \quad \lim_{\eta \to 0} \frac{dk}{d\eta} = \frac{1}{c_e}, \quad \lim_{\eta \to \infty} \frac{dk}{d\eta} = \frac{1}{c_f}.$$

In Fig. 2 the straight lines 1-3 have slopes $1/c_{cr}$, $1/c_e$, $1/c_f$, respectively.

The equilibrium speed of sound $c_e = \sqrt{c_f^{-2} + \int\limits_0^\infty \frac{N(\omega)\, d\omega}{\omega^5}}$ is the maximum possible propagation velocity of low-frequency disturbances. The speed c_{cr} (in the monodispersed case it is equal to zero) is the minimum possible group velocity of low-frequency disturbances. If $c \in (0, c_{cr}) \cup (c_e, c_f)$, then there exists a single stationary point of the phase Φ corrseponding to high-frequency waves. If $c \in [c_{cr}, c_e]$ then there are two stationary points (one corresponding to low-frequency waves and the other to high-frequency waves). Since the dispersion curves are convex, the stationary points of the phase Φ are not inflection points, i.e., the second derivative of Φ with respect to η is nonzero at these points. Hence the solution p(t, x) falls off as $t^{-1/2}$ along the ray $x/t = c$ (see [22], for example).

It is obvious physically that dissipation will lead to a rapid attenuation of high-frequency disturbances. Hence an observer moving with velocity c outside the interval (c_{cr}, c_e) moves practically with respect to the state of rest at large t. Most of the disturbance is localized inside the sector formed by rays with slopes c_{cr} and c_e.

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SEDIMENT TRANSPORT BY TURBULENT FLOW ABOVE A BOTTOM SUBJECT TO EROSION

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The theory of the motion of suspended particles in a turbulent flow at low concentration is presented in [1, 2]. In [3] it is proposed that Coulombic dry friction between the solid particles moving in the liquid be taken into account. In [4-7] the motion of a mixture of a liquid and solid particles is investigated with the help of a rheological relation in the form of a combination of dry friction for the solid phase and viscous friction for the liquid phase. In [4] one-dimensional turbulent flow above an even bottom is considered. In [5-7] the motion is studied in a general formulation with an arbitrary bottom relief and an expression is derived for the sediment flow rate. In [4-7] the particle concentration in the layer of sediment at the bottom is assumed to be constant.

In the present paper we propose, on the basis of the results enumerated above, a model of the medium which gives a continuous description of the motion of the mixture over the entire thickness of the flow, starting from the eroding bottom surface with the limiting particle concentration. Far from the bottom surface, where the concentration is low, the equations convert into the equations of motion derived in [1, 2] for suspended particles in a turbulent flow. The main result is an analytic expression for the sediment flow rate in a turbulent flow for the general three-dimensional problem. The theory does not require the introduction of unknown empirical parameters.

1. Assumptions. We consider the turbulent flow of a heavy incompressible liquid with solid particles in the region $\xi(x, y) < z < \eta$, where x, y, and z is a Cartesian coordinate system whose z-axis is oriented vertically, the equation of the free surface is $z = \eta$, and the equation of the bottom surface is $z = \xi(x, y)$. A stationary granulated uniform medium occupies the region $z < \xi(x, y)$. Mass transfer occurs at the interface $z = \xi(x, y)$. The density of solid particles ρ_p is higher than the density of the liquid ρ_w.

It is assumed that the main mass of the particles moves in a bottom layer of thickness of the order a, much less than the depth $h = \eta - \xi$. The characteristic horizontal since L

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